THESIS PROJECT FOR THE CANDIDATE PHD STUDENT

by

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Abstract. — This project is inspired by the possibility of a geometrization of the p-adic local Langlands correspondence, a perspective which got strong impulsion after recent breakthroughs of Fargues–Scholze for the geometrization of the ℓ-adic local Langlands correspondence. Furthermore, recent work of Emerton–Gee give a natural candidate for the space of Langlads parameters in the p-adic setting, and this space has been further studied by Le–Le Hung–Levin–Morra (LAGA) to capture its finer geometric properties.

The arithmetic p-adic local Langlands correspondence has been proven to be far more complicated when passing from $\mathrm{GL}_2(\mathbb{Q}_p)$ to any other more general group.

The work of the candidate PhD student is focused on a further investigation of the geometry of the Emerton–Gee stack of p-adic Langlands parameters, with a hope to overcome the difficulties appearing beyond $\mathrm{GL}_2(\mathbb{Q}_p)$ by geometric tools coming from the Emerton–Gee approach combined with perfectoid techniques.

1.1. Prolegomena. — The starting point of arithmetic geometry can be identified in the *arithmetica* by Diophantus of Alexandria, and the quest of finding the solutions to polynomial equations in rational coefficients. There is little hope to find uniform, algorithmic procedures to solve the problem (even though this was believed to be possible by Cardano in his *Artis Magnae*) and it is by far more effective to study the *symmetries* of the set of solutions, encoded in Galois groups of number fields, and its representations.

One of the deepest insight in modern arithmetic is the characterization, in simple and exhaustive terms, of the Galois representations arising from geometry. An example of this is the celebrated statement of the Shimura–Taniyama–Weil conjecture, asserting that every elliptic curve over $\mathbb{Q}$ (which gives rise to a Galois representation arising from geometry) is modular (hence, has favorable properties). A far reaching generalization of this is the Fontaine–Mazur conjecture, whose proof in the case of GL_2 over $\mathbb{Q}$ has been among the greatest achievements of the p-adic local Langlands program.

At the heart of the proof of the Shimura–Taniyama–Weil conjecture is the study of *congruences* among automorphic forms. This requires the creation of algebraic and integral structures on objects in non-abelian harmonic analysis (automorphic representations) and of their symmetries (the Hecke algebras), and understand them in Galois theoretic term (i.e. local Langlands correspondences). The crystallization of this approach, the Taylor–Wiles method (generalized by Kisin), gives a mechanism to piece spaces of automorphic forms into coherent sheaves over (product) deformation spaces of local p-adic Galois representations. In this way, most of the difficulty is translated into studying the geometry of these patched sheaves. For instance, understanding the nature of the patched completed homology (i.e. automorphic forms with "infinite level at p") is at the core of the p-adic local Langlands correspondence, which is established for $\mathrm{GL}_2(\mathbb{Q}_p)$ (and GL_1), but is very mysterious in any other cases.

The above setting happens at the level of formal geometry, i.e. on deformation spaces of a fixed residual Galois representation $\overline{\rho}$. However, recently, Emerton and Gee have shown how to meaningfully vary $\overline{\rho}$: they construct the Emerton–Gee stack $\mathcal{X}^{EG}_{d,K}$ which is a moduli stack of all rank d continuous representations of G_K, where K is a p-adic field (i.e. the "p-adic Langlands parameters"). In particular the previous deformation spaces are recovered as completions of $\mathcal{X}^{EG}_{d,K}$ at closed points. It is thus natural to hope for a geometrization of the p-adic local Langlands correspondence over the Emerton-Gee stack, in analogy with the geometrization of the ℓ-adic local Langlands correspondence ($\ell \neq p$) in the recent breakthrough work of Fargues–Scholze.

As elucidated above, the p-adic local Langlands correspondence is extremely mysterious outside $\mathrm{GL}_2(\mathbb{Q}_p)$. However, the recent results of [**BHH**$^+$] shed some hint on its nature for $\mathrm{GL}_2(K)$ where $K/\mathbb{Q}_p$ is unramified. These results can be understood as exhibiting certain "pointwise" properties of the universal family of $\mathrm{GL}_2(K)$-representation living over $\mathcal{X}^{EG}_{2,K}$. The PhD project of the candidate PhD student will be in the context of interpolating such information in families, and in particular creating the necessary foundations for such investigations.

1.2. Detailed description of the PhD project of the candidate PhD student. — As mentioned above, Emerton and Gee succeed in defining a moduli $\mathcal{X}^{EG}_{d,K}$ of p-adic local Langlands parameters [**EG**] (here, K is a finite extension of $\mathbb{Q}_p$ which we assume to be unramified). The key insight is that while one can not make reasonable family of p-adic Galois representation outside a purely formal setting (i.e. over Artinian or complete local bases), one can make algebraic families of *étale* (φ,Γ)*-modules*. Yet, over complete local bases, Fontaine's equivalence from p-adic Hodge theory shows that the two concepts coincide. For this reason, $\mathcal{X}^{EG}_{d,K}$ is constructed as the moduli of (projective rank d) étale (φ,Γ)-modules.

Among the spectacular success of the Emerton–Gee approach is their proof of the existence of crystalline lifts for a mod p Galois representation $\overline{\rho} : G_K \overset{\text{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \twoheadrightarrow \mathrm{GL}_n(\mathbb{F})$. The existence of such lifts depends on the vanishing of a certain $H^2(G_{\mathbb{Q}_p},\overline{\rho})$, which was already known when the *inertial weights* of $\overline{\rho}$ were in a sufficiently generic position. Emerton and Gee are able to interpolate these cohomology groups into a vector bundle $\mathcal{H}$ over $\mathcal{X}^{EG}_{d,K}$. By geometric considerations on its support at closed points (ultimately considerations on the dimension of the tangent space of $\mathcal{X}^{EG}_{d,K}$), and the vanishing of $\mathcal{H}$ at "generic" points, Emerton–Gee prove its vanishing at *all* points, hence the existence of crystalline lifts for any $\overline{\rho}$.

The above example is an illustration of the usefulness of doing things in families: one can deduce pointwise information from "easy" points and investigating the geometry of the family.

The fact that the Emerton–Gee stack is constructed in terms of étale (φ,Γ)-modules also makes it germane to the p-adic local Langlands. Indeed, at the core of the construction of p-adic local Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$ is Colmez's functor, which produces a (φ,Γ)-module out of a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. However when $\mathbb{Q}_p$ is replaced by K, the representation theory of $\mathrm{GL}_2(K)$ becomes very complicated, and there is no naive generalization of Colmez's recipe. In fact, in this settings there are various flavors of (φ,Γ)-modules: some prominent examples of these are the Lubin–Tate (φ,Γ)-modules (where $\Gamma = \mathcal{O}_K$ and the descent is via the Lubin-Tate extension instead of the cyclotomic extension), and the multvariable (φ,Γ)-modules of Carter–Kedlaya–Zabradi [**CKZ**]. The first produce an equivalence with p-adic representations of G_K, while the second an equivalence with representation of a product of $[K:\mathbb{Q}_p]$ copies of $G_{\mathbb{Q}_p}$. There is also a third kind, which seems as a hybrid of the Lubin-Tate and the multivariable versions that recently plays a prominent role in [**BHH**$^+$].

In light of the above discussion, the natural problem is to construct moduli spaces of the various flavors of (φ,Γ)-modules above, and establish the relations between them as predicted by a "p-adic local Langlands in families".

In particular, the stack of (rank d) Lubin–Tate (φ,Γ)-modules, call it $\mathcal{X}^{LT}_{d,K}$, should exist and be isomorphic to the usual $\mathcal{X}^{EG}_{d,K}$, simply because the usual and Lubin–Tate (φ,Γ)-modules are both equivalent, over Artinian bases, to G_K-representation.

As a warm-up question for the candidate PhD student would be to the construction of the stack $\mathcal{X}^{LT}_{d,K}$. *While this should be straightforward from the Emerton–Gee's setup, this would help Pham to familiarize with basic concepts of integral* p*-adic Hodge theory, and adapt the cyclotomic descent from* $\mathbf{A}_{\mathrm{inf}}$ *performed by Emerton–Gee for the Lubin–Tate descent. Furthermore, Pham should be able to construct a morphism of stack from* $\mathcal{X}^{LT}_{2,K}$ *to* $\mathcal{X}^{EG}_{2^{[K:\mathbb{Q}_p]},\mathbb{Q}_p}$ *(and analogously in higher dimension) which in Galois theoretical terms is given by the tensor induction.*

The objects introduced so far are (families of) projective modules (endowed with a Frobenius structure and a descent by either the cyclotomic or Lubin–Tate tower) over the p-adic completion of $\mathcal{O}[\![X]\!][1/X]$, where $\mathcal{O}$ is $\mathbb{Z}_p$ or $\mathcal{O}_K$ in the cyclotomic and Lubin–Tate case respectively. On the other hand, the various flavors of multivariable (φ,Γ)-modules (such as those in [**CKZ**] and [**BHH**$^+$]) concerns (families of) projective modules over a completed tensor products of p-adic completion of $\mathcal{O}[\![X]\!][1/X]$ This innocuous change leads to a host of technical complications. In the case of $\mathcal{X}^{EG}_{d,K}$, one has to first consider a moduli of various *integral* versions of (φ,Γ)-modules in order show that $\mathcal{X}^{EG}_{d,K}$ is reasonable (i.e. an ind-algebraic stack): one first has to study moduli of Breuil–Kisin(–Fargues) modules of finite heights. Furthermore, the study of such moduli spaces naturally lead to various closed substack of $\mathcal{X}^{EG}_{d,K}$ which are related to p-adic Hodge theoretic conditions (e.g. crystalline). Finally, Emerton–Gee also gives a parametrization of the irreducible components of the underlying reduced stack of $\mathcal{X}^{EG}_{d,K}$ in terms of representation theoretic data (the Serre weights), and the Breuil–Mézard conjecture gives a representation theoretic description of the cycles of the crystalline substacks.

The second step in Pham's PhD work will be to construct a moduli stack $\mathcal{X}^{\mathrm{CKZ}}_{d,\mathbb{Q}_p}$ *of Carter–Kedlaya–Zabradi* (φ,Γ)*-modules and study its fundamental properties, as ind-algebraicity, dimension, the dimension and parametrization of its irreducible components. In particular, along the way, he will need to investigate multivariable analogues of Breuil–Kisin(–Fargues) modules. As a bonus, such investigation might lead to the right multivariable version of* p*-adic Hodge theory conditions (such as crystalline etc.), and he should study the various closed substack cut out by such conditions. He should also investigate whether there are any representation theoretic interpretation of the cycles of these multivariable crystalline stubstacks (i.e. whether there can be a "multivariable Breuil–Mézard").*

As mentioned above, one of the motivations for studying the various multivariable versions of (φ,Γ)-modules is that they seem to arise more naturally from representations of p-adic groups when $K \neq \mathbb{Q}_p$. The interpolation of the results of [**BHH**$^+$] and the existence of the hypothetical universal family of representations of $\mathrm{GL}_2(K)$ over the appropriate Emerton–Gee stack suggest that there should be maps producing multivariable (φ,Γ)-modules out of single variable ones, which is not obvious!

In the third part of his PhD project, Pham should investigate the possibility of constructing maps between the various stacks of multivariable and single variable (φ,Γ)*-modules in the above discussion, and investigate their images.*

The above problems are motivated primarily by the p-adic local Langlands for GL_2 (and GL_n). However, even for GL_1 (where the correspondence is clear by local Class Field Theory) there are interesting problems related to the geometrization. In the setting of the de Rham geometric Langlands program, Laumon used a generalized Fourier-Mukai transform to establish a categorical Langlands correspondence for GL_1, giving an equivalence between quasi-coherent sheaves on the stack of line bundles with a flat connection, and D-modules on the Picard stack (of a curve). It would be interesting to investigate a p-adic analogue of that, where the role of the stack of line bundles with a flat connection would be played by the Emerton–Gee stack for GL_1 and the Picard stack is now the Picard stack for the Fargues–Fontaine curve.

In the ℓ-adic setting (which is much closer to Betti geometric Langlands) Zhu made a precise conjecture in [**Zhu**] for tori (Conjecture 3.2.2 in *loc. cit.*, subsequently expanded by Fargues and Scholze) relating quasi-coherent sheaves on the stack of T-valued Langlands parameters with the Picard groupoid of ${}^cT$-torsors over $\breve{\mathbb{Q}}_p$ which are Frobenius stable.

A problem for the candidate PhD student is to describe the category of coherent sheaves on the Emerton–Gee stack for GL_1*, and relate it to some category on the Picard stack of the Fargues–Fontaine curve*

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