Sujet proposé par Jérôme Le Rousseau et Thomas Duyckaerts

RESEARCH PROJECT: CONTROLLABILITY FOR NONLINEAR WAVE EQUATION

This research project concerns the control theory of the semilinear wave equation, which appears in many physical models. This problem has already been treated for some types of nonlinearity (see references below). The goal of this project is to treat the cases of supercritical and focusing nonlinearities, which are almost completely open. We would like in particular to apply the methods developed recently to study the dynamics of focusing wave equations (see e.g. [6]) to the control theory of these equations.

We first review aspects in the case of a linear wave equation. An important property is the *observability* in time T, that is the fact that one can estimate a solution, only from the knowledge of this solution in a particular region of the space during a time interval of length T. This observability property can be of great importance for imaging techniques when one wishes to understand properties of a the inside of a body by probing it with waves. One can think of medical imaging at the scale of acoustic ultrasound waves yet also of seismic imaging at the scales of elastic waves with wavelength of hundreds of meters or more.

For linear wave equations, high-frequency information is known to travel along rays of geometrical optics. Hence, if one wishes to observe a wave in some time interval of length T one necessarily needs to require the following *geometric control condition*: every possible ray has to enter the region of observation within that time T after its initiation. A major result in the early 90's is that of C. Bardos, G. Lebeau and J. Rauch: the previous geometrical condition is not only necessary, it is also sufficient for the observability of wave in time T. The true "tour de force" of Bardos-Lebeau-Rauch was their ability to obtain this result even in the presence of a boundary on which rays of geometrical optics bounce. Even then, the geometrical condition fully captures high-frequency information on the solution; this property is now better understood by the use of semiclassical measures associated with sequences of solutions and concentration phenomena. Such measures get transported along the bicharacteristic flow, with reflections at boundaries, or more complex phenomena depending on how a ray reaches or leaves the boundary: transversally, tangentially... at various orders. Furthermore, the geometrical condition fully captures low-frequency information; this is based on unique continuation arguments for the wave equation.

Once observation for waves is understood, this property naturally translates into exact controllability properties, that is, the ability to start from any state and to reach any other prescribed state by only acting on the solution in the observation region, called "control region" in that framework. Equivalent is also the stabilization of the wave equation by means of a localized damping term. Assuming the geometric control condition, one proves that the energy of the waves decays exponentially, a remarkable property if compared to the default logarithmic decay obtained if damping occurs in a region that does not satisfy the geometrical condition described above. The exact controllability property can be important if one tries to achieve a particular state for a wave, for instance focusing energy in a particular point. Damping and fast energy decay can also be essential; think of an elastic structure like an airplane wing: stabilization in the case of oscillation sounds like a property to look forward to.

In this project, we would like to consider *nonlinear* wave equations, that is add to the linear wave equation mention aboved a nonlinearity, which is typically a power of the solution. The equation has a conserved energy. Depending on the sign of the nonlinearity, the equation is said to be defocusing or defocusing. In the defocusing case, the terms in the energy all have the same sign and the energy gives a good control of the solution. Blow-up in finite time is usually excluded. In the focusing case, the nonlinear term can provoke blow-up in finite time. The power of the nonlinearity is also relevant, the energy is said to be (energy) subcritical/critical/supercritical if its power is below/equal to/above a certain threshold linked to the scaling of the energy. The energy conservation law is effective in the subcritical case, and almost useless in the supercritical one.

The controllability of nonlinear wave equation was mainly considered for energy-subcritical, defocusing nonlinearities. In this case it has been proved in [8] (see also [4], [3], [7]) that the GCC implies the controllability. Note that [8] considers much more general nonlinearities.

The defocusing, critical case was treated in [10], where controllability was proved under a condition slightly stronger than the geometrical condition and a smallness condition on the low-frequency of the data.

Very few is known when the nonlinearity is focusing. Indeed, only the related *stability* problem when $\Omega = \mathbb{R}^n$ was treated in [1] (see also [2]). As observed in [11], it follows from finite speed of propagation that the geometrical condition is not sufficient for controllability: if a solution of the equation without control blows up in finite time T, and if T is smaller than the time given by the geometric control condition, then controllability is impossible for this initial data.

In this research project we would like to consider the controllability for the nonlinear wave equation in cases that are not treated by the works cited above, and in particular when the nonlinearity is focusing. It is reasonable to conjecture that if the geometric control condition is satisfied in time T, and the solution without control is global or blows up in finite time T' larger than T, then the solution is controllable to 0.

A first step of this project would be to consider the case of one space dimension, and a second step would be to consider a bounded domain in higher dimension, with a stronger condition than the geometric control condition (typically a geometrical condition such that multipliers method are available). This can be done first for subcritical nonlinearities (in the continuity of the master internship of the applicant, concerning the article [7]), then for critical (and if possible, supercritical) nonlinearities. An other interesting problem would be to consider, as in [1], the stabilization property. The strong condition in [1] on the initial data excludes the blow-up in finite time, and we would like to treat more general cases in accordance with our conjecture. Another related problem (linked to the internship of the applicant with Laurie Oksanen) is the inverse problem for nonlinear wave equations.

As mentioned before, one goal of this project is to adapt methods that have been developed in the study of the dynamics of the nonlinear wave equation in the setting of control theory. These methods include compactness/rigidity argument (see [9]), the method of channels of energy (see e.g. [6]) and generalized Morawetz estimate as in [5]).

Research plan.

(1) First months: control and stability in space dimension 1, generalization of the results of [11].
(2) End of the first year and second year: control and stability of focusing, subcritical wave equations in higher space dimensions (following references [1], [7], [8]).
(3) Third year: critical nonlinearity (generalization of [10] to the focusing case). Supercritical case and/or inverse problems.

References

[1] Lassaad Aloui, Slim Ibrahim, and Kenji Nakanishi. Exponential energy decay for damped Klein–Gordon equation with nonlinearities of arbitrary growth. *Communications in Partial Differential Equations*, 36(5):797–818, 2010.

[2] Nicolas Burq, Geneviève Raugel, and Wilhelm Schlag. Long time dynamics for damped Klein-Gordon equations. *Ann. Sci. Éc. Norm. Supér. (4)*, 50(6):1447–1498, 2017.

[3] Belhassen Dehman and Gilles Lebeau. Analysis of the HUM control operator and exact controllability for semilinear waves in uniform time. *SIAM Journal on Control and Optimization*, 48(2):521–550, 2009.

[4] Belhassen Dehman, Gilles Lebeau, and Enrique Zuazua. Stabilization and control for the subcritical semilinear wave equation. In *Annales scientifiques de lEcole normale supérieure*, volume 36, pages 525–551. Elsevier, 2003.

[5] Thomas Duyckaerts, Hao Jia, Carlos Kenig, and Frank Merle. Soliton resolution along a sequence of times for the focusing energy critical wave equation. *Geometric and Functional Analysis*, 27(4):798–862, 2017.

[6] Thomas Duyckaerts, Carlos Kenig, and Frank Merle. Classification of radial solutions of the focusing, energy-critical wave equation. *Cambridge Journal of Mathematics*, 1(1):75–144, 2013.

[7] Romain Joly and Camille Laurent. Stabilization for the semilinear wave equation with geometric control condition. *Analysis & PDE*, 6(5):1089–1119, 2013.

[8] Romain Joly and Camille Laurent. A note on the semiglobal controllability of the semilinear wave equation. *SIAM Journal on Control and Optimization*, 52(1):439–450, 2014.

[9] Carlos E. Kenig and Frank Merle. Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation. *Acta Math.*, 201(2):147–212, 2008.

[10] Camille Laurent. On stabilization and control for the critical Klein–Gordon equation on a 3-d compact manifold. *Journal of Functional Analysis*, 260(5):1304–1368, 2011.

[11] Enrique Zuazua. Exact controllability for semilinear wave equations in one space dimension. In *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, volume 10, pages 109–129. Elsevier, 1993.

References

[1] Lassaad Aloui, Slim Ibrahim, and Kenji Nakanishi. Exponential energy decay for damped Klein–Gordon equation with nonlinearities of arbitrary growth. *Communications in Partial Differential Equations*, 36(5):797–818, 2010.

[2] Nicolas Burq, Geneviève Raugel, and Wilhelm Schlag. Long time dynamics for damped Klein-Gordon equations. *Ann. Sci. Éc. Norm. Supér. (4)*, 50(6):1447–1498, 2017.

[3] Belhassen Dehman and Gilles Lebeau. Analysis of the HUM control operator and exact controllability for semilinear waves in uniform time. *SIAM Journal on Control and Optimization*, 48(2):521–550, 2009.

[4] Belhassen Dehman, Gilles Lebeau, and Enrique Zuazua. Stabilization and control for the subcritical semilinear wave equation. In *Annales scientifiques de lEcole normale supérieure*, volume 36, pages 525–551. Elsevier, 2003.

[5] Thomas Duyckaerts, Hao Jia, Carlos Kenig, and Frank Merle. Soliton resolution along a sequence of times for the focusing energy critical wave equation. *Geometric and Functional Analysis*, 27(4):798–862, 2017.

[6] Thomas Duyckaerts, Carlos Kenig, and Frank Merle. Classification of radial solutions of the focusing, energy-critical wave equation. *Cambridge Journal of Mathematics*, 1(1):75–144, 2013.

[7] Romain Joly and Camille Laurent. Stabilization for the semilinear wave equation with geometric control condition. *Analysis & PDE*, 6(5):1089–1119, 2013.

[8] Romain Joly and Camille Laurent. A note on the semiglobal controllability of the semilinear wave equation. *SIAM Journal on Control and Optimization*, 52(1):439–450, 2014.

[9] Carlos E. Kenig and Frank Merle. Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation. *Acta Math.*, 201(2):147–212, 2008.

[10] Camille Laurent. On stabilization and control for the critical Klein–Gordon equation on a 3-d compact manifold. *Journal of Functional Analysis*, 260(5):1304–1368, 2011.

[11] Enrique Zuazua. Exact controllability for semilinear wave equations in one space dimension. In *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, volume 10, pages 109–129. Elsevier, 1993.