PhD title: "Box-integrality and the integer decomposition property"

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Scope. A *lattice polytope* in $\mathbb{R}^d$ is the convex hull of a finite number of points in the lattice $\mathbb{Z}^d$. Recently, Chervet, Grappe, and Robert [5] introduced a class of lattice polytopes with strong integrality properties: fully box-integer polytopes. A lattice polytope is *box-integer* when intersecting it with any integer box yields a lattice polytope again, and *fully box-integer* when all its integer dilations are box-integer. These polytopes play a fundamental role in combinatorial optimization because they are behind strong min-max theorem, such as the MaxFlow-MinCut theorem of Ford and Fulkerson [8].

The goal of the thesis is to study two decomposition properties of fully box-integer polytopes. There are two ingredients to approach these polytopes, that were not investigated previously and seem promising: using the matricial characterizations given in [5], or studying the toric varieties associated to them [1].

A lattice polytope P has the *integer decomposition property* (IDP) if for every positive integer k, every integer point in kP is the sum of k integer points in P. Since Oda [14] introduced the question to identify the polytopes that have the IDP, these polytopes have been of interest in integer programming, graph theory, commutative algebra, and toric geometry.

It turns out that the abstract class introduced by Gijswijt and Regts [9], a class that has the IDP, is a subclass of fully box-integer polytopes. Unfortunately, not all fully box-integer polytopes have the IDP. The first goal of the thesis is to determine which ones do. A first step would be a subcase of Oda's question:

Open problem 1 *Is it true that every smooth fully box-integer polytope has the integer decomposition property?*

A full-dimensional polytope of $\mathbb{R}^d$ is *simple* if every vertex has n neighbors. A simple integer polytope is *smooth* if for every vertex v the generators of the associated minimal tangent cone form a basis of the lattice $\mathbb{Z}^d$. Equivalently, the toric variety associated to the polytope is smooth. In dimension three, this problem is settled if the polytope is moreover centrally symmetric [4], but open in general. Nothing is known in dimension $d \geq 4$.

A stronger property than IDP is the Integer Carathéodory Property (ICP): a lattice polytope P has the ICP if for every positive integer k and integer point x in kP, there exist positive integers $n_1, \ldots, n_t$ and affinely independent integer points $x_1, \ldots, x_t$ in P such that $x = n_1 x_1 + \cdots + n_t x_t$. The second goal of the thesis is to study which fully box-integer polytopes have the ICP. In particular, the r-arboresence polytope [15, p. 182] is fully box-integer, hence it contains the following open problem [16].

Open problem 2 *Does the r-arborescence polytope have the Integer Carathéodory Property?*

Polytopes have also been playing a key role in the development of homotopical algebra ever since the foundation of this theory [17]. This domain, prompted by the problem of recognizing loop spaces in algebraic topology, gave rise to rich algebraic structures like the notion of an associative algebra up to homotopy (aka A_∞-algebra) and the notion of an operad [12]. Nowadays, these higher structures have a ubiquitous foundational role in many branches of mathematics (algebraic topology, algebraic geometry, mathematical physics, non-commutative probability) and they have diffused in computer science as well (combinatorics, category theory, logic, rewriting systems).

Quite surprisingly, lattice realisations of families of operadic polytopes were recently needed in order to develop further the algebraic properties of the aforementioned algebraic structures. in [13], Masuda, Thomas, Tonks, and Vallette considered the lattice realisations of the associahedra due to Loday [11] in order to settle a universal formula for the tensor product of A_∞-algebras. Laplante-Anfossi, a former PhD student of Vallette, went further and considered in [10] lattice realizations of the operahedra to settle a universal formula for the tensor product of homotopy operads. As mentioned above, the toric varieties associated to lattice polytopes are very interesting objects to study. For instance, Dotsenko, Shadrin, and Vallette studied in [7] the properties of the toric varieties associated to Loday's realisation of the associahedra. In this direction, the third part of the thesis will focus on the following open problem.

Open problem 3 *Study the integer decomposition property and the toric varieties for the families of operadic polytopes.*

Interactions. These topics impact both the research interests of the "Algorithmes et Optimisation Combinatoire" team of the LIPN and of the "Topologie Algébrique" team of the LAGA. Indeed, fully box-integer polytopes already form an important part in the research activities of the AOC team, and Roland Grappe published a number of papers on this topic [2, 3, 5, 6]. Operadic lattice polytopes and their associated toric varieties lie whithin the research directions of the TA team since Bruno Vallette already worked on these topics in [7, 13].

We have been identifying these common fields of study for several years now; Unfortunately we were not able to find enough free time to develop a deep and meaning collaboration so far. **In this way, the present join Ph.D. project will be an extremely fruitful way to create interesting and promising interactions between these two teams and these two departments.**

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