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Proposition de sujet de thèse :

On the representations of some classes of Lie algebras of vector fields

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Lie algebras of vector fields have been a constant subject of interest since the end of the 19th century following the fundamental works of Sophus Lie and Elie Cartan.

Let A be a finitely generated algebra over the (algebraically closed field $\mathbb{K}$ of caracteristic 0). Denote by $\mathrm{Der}_{\mathbb{K}}(A)$ the set of derivations of A. Then $\mathrm{Der}_{\mathbb{K}}(A)$ is a Lie subalgebra of $\mathrm{End}_{\mathbb{K}}(A)$; it will be called the Lie algebra of vector fields associated to the (algebraic variety corresponding to the) algebra A. The Lie algebra $\mathrm{Der}_{\mathbb{K}}(A)$ is simple if and only if the corresponding variety is smooth and irreducible [BF1], hence we have a large supply of examples of (infinite dimensional) simple Lie algebras. We observe that these Lie algebras are significantly different from the standard Lie algebras of vector fields in the C^∞ case (the latter Lie algebras are not simple).

The main problem of this theory is to classify simple $\mathrm{Der}_{\mathbb{K}}(A)$-modules. In the case where the variety is just the affine space $\mathbb{K}^n$, we get the Lie algebra $\mathcal{V}_n$ of so-called polynomial vector fields. Important results for representations of this algebra where obtained in the 1970's, [Ru]. Another iconic example is that of the n-th Witt algebra W_n, where $W_n = \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}])$. The corresponding variety is the n-dimensional torus. The Lie algebra W_1 is tightly linked to the celebrated Virasoro Lie algebra. If $\mathcal{V}$ is one of these Lie algebras of vector fields then there exists a relevant Cartan subalgebra $\mathcal{H}$ of $\mathcal{V}$, which is an abelian Lie subalgebra. If M is a representation of $\mathcal{V}$, one can ask whether the elements of $\mathcal{H}$ act on M by simultaneously diagonalisable endomorphisms. If this is the case, M is called a *weight representation* and, if for $\lambda \in \mathcal{H}^*$ we put $M^\lambda = \{v \in M \,|\, \forall h \in \mathcal{H},\, h.v = \lambda(h)v\}$, then M is the direct sum of the weight spaces M^λ. Then, a typical attempt consists in classifying Harish-Chandra representations, that is those which are weight representations with finite dimensional weight spaces. The interest in the weight representations must be understood in the light of the case of complex finite dimensional semisimple Lie algebras for which *all* the finite dimensional representations are weight representations.

Simple weight Harich-Chandra modules for the Witt algebras were classified in [BF], generalizing a well-known theorem of Mathieu [Ma] on the classification of simple weight modules for the Virasoro algebra. For polynomial Witt algebras such classification was obtained in [GS].

Until recently essentially nothing was known for general algebraic varieties. The main difficulty is that the classical methods of Lie theory are not applicable in this case. First results were obtained in the papers [BFN], [BN], [BNZ]. In particular, two new families were constructed for an arbitrary irreducible affine variety X - so-called gauge modules and Rudakov modules. They do not exhaust all simple modules but at least they cover all known classes of representations. A key ingredient of the theory is a notion of AV-modules, those that admits compatible actions of the Lie algebra of vector fields and the commutative algebra of functions on the variety X.

Main objective of the project is contribute to the representation theory of Lie algebras of algebraic vector fields on general affine algebraic varieties. Namely, we plan to construct new important classes of representations of these Lie algebras by relaxing certain restrictions in the constructions of gauge and Rudakov modules and describe their structure. We will focus on the cases of n-dimensional torus, n-sphere and algebraic curves.

Also, quantizations of these Lie algebras and representations will be considered. The theory of quantum groups provides natural and important examples of noncommutative algebraic varieties. It will be interesting to explore wether the methods applied in the above classical setting may be extended to this quantum setting. It seems that only the case of quantum tori was extensively studied at the moment, [LT], [LZ].

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