ÉCOLE DOCTORALE GALILÉE
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Titre du sujet : Polymères aléatoires et processus renforcés

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This Ph.D. project lies in the domain of probability theory and statistical mechanics. The main goal is to explore and exploit the relations between two models that have each attracted significant attention in the past decades: the directed polymer in random environment and the Vertex-Reinforced Jump Process. These two models have been the object of intense research activity in recent years, not only because their intrinsic interest but also due to their connections to several relevant questions in probability and statistical mechanics. On the one hand, the directed polymer model is closely related to the question of disorder relevance in physics systems, and to some stochastic PDEs such as the stochastic heat equation and the KPZ (Kardar-Parisi-Zhang) equation. On the other hand, the Vertex-Reinforced Jump Process is related to other reinforced stochastic processes (such as the Edge-Reinforced Random Walk), the H2|2 supersymmetric hyperbolic sigma model and to a random Schrödinger operator.
Let us briefly present the two models before turning to their connections and the questions we propose to tackle during the Ph.D.
Directed polymers in random environment. The directed polymer model was introduced 40 years ago by Huse and Henley [14], and has since then been studied in depth. It can be thought of as a simplified model for a polymer placed in some inhomogeneous medium. The two main ingredients of the mathematical model are a simple random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}^d$, whose time-space trajectory $(n, S_n)_{n\geq 0}$ is interpreted as a directed polymer, and a family of time-space independent and identically distributed (i.i.d.) random variables, assumed to be centered, referred to as a random environment. The model consists of modifying the law of the simple random walk up to time N by a Boltzmann-Gibbs weight, with Hamiltonian given by the sum of the random variables sitting on the time-space random walk trajectory. The model has a natural parameter $\beta \geq 0$ which is the inverse-temperature, and the goal is to understand the long-time behavior of the random walk paths under the Gibbs measure, depending on the parameter β. One may refer to the monograph [8] for a complete overview of the model, and to [24] for an account of the recent developments.
In a nutshell, it has been shown that the model exhibits a phase transition. There exists some critical inverse-temperature β_c such that: if $\beta < \beta_c$ then the polymer trajectories have diffusive trajectories (in fact, an invariance principle holds); whereas if $\beta > \beta_c$, the trajectories exhibit some

localization property, somehow concentrating in some narrow corridors where the environment is favorable. A observation, due to Bolthausen [5] is that the partition function, that normalizes the Gibbs measure to a probability measure, is a martingale for the natural time-filtration. The critical point β_c can then be interpreted as the critical point for the martingale to vanish in the 1large N limit, and the rate of convergence to zero is related to some localization properties of the paths, see [9].

To summarize a series of works ([5, 7, 9, 10, 17]), the critical point is shown to be $\beta_c = 0$ in dimension $d = 1, 2$ (this is an instance of disorder relevance), whereas $\beta_c > 0$ in dimension $d \geq 3$ (disorder irrelevance). The sharpness of the phase transition, that is the fact that the martingale decreases exponentially fast for any $\beta > \beta_c$, has been proved in dimension $d = 1, 2$ in [10, 17], but it is only very recently that it has been proven in dimension $d \geq 3$, see [15].

Another line of recent research has been to investigate the so-called intermediate disorder regime, and its relation to the stochastic heat equation, since the partition function of the directed model can be interpreted as a discrete version of it. Indeed, choosing a proper normalization for $\beta = \beta(N)$, it has been shown that in dimension $d = 1, 2$, the directed polymer model possesses a non-trivial scaling limit, intimately related to the stochastic heat equation, see [1] for dimension $d = 1$ and [6] for dimension $d = 2$ (the latter giving a first interpretation for a solution of the 2d stochastic heat equation).

The vertex reinforced jump process. The Vertex-Reinforced Jump Process (VRJP) is a model proposed by Werner, which was first investigated by David and Volkov [11]. It considers a continuous-time random walk on a graph — let us take the simple random walk on $\mathbb{Z}^d$ to simplify the exposition —, whose jump rates evolve with time and are reinforced in the following way. At a given time t, the jump rate from a site to a neighboring vertex x is proportional to $W(1 + \ell_t(x))$, where $W > 0$ is a parameter of the model and $\ell_t(x)$ denotes the current local time of the walk at vertex x. Put otherwise, a vertex that has been frequently visited in the past (i.e. with $\ell_t(x)$ large) makes the random walk more likely to visit it in the future, leading to a reinforcement mechanism. The parameter W can be thought as modulating the strength of the reinforcement: if W is large, the VRJP spends a little time at the vertices, and the relative weight of the local time is negligible compared to 1 (weak reinforcement regime), whereas if W is small, the relative weight of the local time is important (strong reinforcement regime). Then, the goal is to obtain long-time properties of the Vertex-Reinforced Jump Process, in particular regarding its recurrent or transient behavior depending on the parameter W .

A recent article by Poudevigne [19] shows some monotonicity properties in W of the model, which seems natural but is not obvious at all from the definition. This, in particular, proves the existence of a critical strength W_c such that the VRJP is recurrent for $W < W_c$ and transient for $W > W_c$. As a consequence of an earlier series of works ([2, 12, 13, 16, 18, 20, 21, 23]) one gets that $W_c = +\infty$ in dimensions $d = 1, 2$, whereas $0 < W_c < +\infty$ in dimension $d \geq 3$; this mirrors the phase transition for the directed polymer model.

In recent papers, Christophe Sabot, Pierre Tarrès and Xiaolin Zeng [22, 23] introduced a Schrödinger operator representation of the VRJP. In a few words, up to a time-change, the VRJP can be interpreted as a random walk in a (specific, W dependent) random potential. In [23], the authors then introduce a martingale that encodes key properties of the VRJP. This martingale converges to zero if and only if the VRJP is recurrent. Moreover, its integrability properties are related to the diffusivity of the VRJP, and its rate of convergence to zero is expected to encode 2some of its localization properties. Hence, the critical strength W_c is the critical point for the martingale to vanish in the large size limit. This mirrors the directed polymer phenomenology, but the state of the art for the VRJP is however much less advanced: for instance, exponential localization for all reinforcement strength W in dimension $d = 2$ or sharpness of the phase transition in dimension $d \geq 3$ remain major open questions.

Relation between the two models and main questions for the Ph.D. project. One important observation is that the VRJP martingale introduced in [23] formally looks like the partition function of a polymer in a random environment, with two major differences from the directed polymer model: (i) polymers are not directed and may visit several times a given site; (ii) the random environment is not i.i.d., but possesses some specific structure. This is the starting point of the recent article [4], where the authors exploit both "polymer" and "VRJP" ideas to obtain some results on the Lp integrability of the VRJP martingale.

The goal of the Ph.D. project is to further develop this polymer approach to the study of the VRJP (and possibly other reinforced processes or Schrödinger operators). In particular, a number of questions have been solved for directed polymers with general techniques, that could prove useful in the context of the VRJP, opening the way to a further understanding of its fine properties. Here are some natural research directions arising from this approach, presented in apparent order of increasing difficulty:

• Relate explicitly the VRJP martingale and its convergence rate to 0 to localization properties of the VRJP, in the spirit of [9];

• Get a better understanding of the so-called L2 regime of the VRJP martingale, where a lot of results have been obtained for the directed polymer, see e.g. [8, Chapter 3];

• Consider an intermediate disorder regime for the VRJP in dimension d = 1 and show some relation with a stochastic PDE (the case of dimension d = 2 should be much more subtle).

Let us also provide two more ambitious questions that the Ph.D. project might lead to if the approach reveals fruitful enough:

• Prove the exponential decay of the VRJP martingale in dimension d = 2, for any reinforcement parameter W < +∞, using either the original approach from [17, 3] (which is by now better understood) or the new ideas emanating from [15];

• Prove the sharpness of the phase transition in dimension d ≥ 3.

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