Proposition d'un sujet de thèse à l'Ecole Doctorale Galilée

Université Sorbonne Paris Nord

Titre du sujet :	Study of the polyhedral links between coloring and matching problems
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Discipline :	Informatique
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Sujet: Given a graph $G = (V, E)$ and a set of colors $\{1, \ldots, k\}$ the coloring problems look for the minimum number k such that incident elements receive distinct colors. Several classical variants of the problem have been extensively studied in the literature. Among them, we recall three important problems. The Vertex Coloring Problem (VCP) where the aim is to assign colors to vertices in such a way that adjacent vertices receive different colors, a formulation that arises naturally in contexts such as scheduling and register allocation [4]. The Edge coloring Problem (ECP), on the other hand, requires that adjacent edges be colored differently, with applications in traffic job scheduling, and sports tournaments [10, 13]. A more general formulation is the Total Coloring Problem (TCP), where both vertices and edges must be colored while avoiding any conflict between adjacent or incident elements. The TCP encompasses the constraints of both VCP and ECP, making it a particularly challenging and rich area of study [11, 8, 2]. Parallel to the family of coloring problems lies another well-studied class of combinatorial problems: matching problems. These aim to select sets of graph elements, vertices, or edges such that no two selected elements are adjacent or incident. Classical examples include the Stable Set Problem [9, 7, 14, 3], and the well-known Maximum Matching Problem [5], both of which have significant implications in graph theory. More recently, the Total Matching Problem [1, 6, 8, 12] offer alternative structural viewpoints that closely align with coloring constraints. Specifically, The Total Matching Problem can be seen as a natural abstraction of Stable Set and Matching Problem, where the goal is to select vertices and edges in such a way that none are adjacent or incident and of maximum size. These exclusion rules, which reflect the constraints of the Total Coloring Problem (TCP), become even more intriguing when examined through a polyhedral lens, revealing deep structural connections between the two problems. The similarities between the underlying polytopes of coloring and matching can help us build tighter formulations.

This PhD project lies right at the intersection of these two areas. On one side, it aims to push forward the understanding of total coloring by studying its polyhedral structure, proposing new integer programming models, and testing these ideas in practical settings like telecommunications, scheduling, and network design. On the other side, it looks at total matching not just as a related problem, but as a potential tool to improve our approach to total coloring, both in theory and in computation. To this aim, several techniques will be studied with both theoretical and computational aspects, including dedicated Branch-and-Cut and Branch-and-Price algorithms for the considered coloring variants. In particular, the hope is to propose better relaxation values on the literature graph benchmark instances.

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