Contrat doctoral – ED Galilée

Titre du sujet : Cohesion, Linear Logic, and Differentiation

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- *Discipline* : Informatique
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- *Domaine de recherche* : logique
- *Mots clés* : logique linéaire, théorie des types, théorie des catégories, topos cohésif

Context: Ehrhard and Regnier's **Differential linear logic (DiLL)** [1, 2] and the **synthetic differential geometry (SDG)** of Lawvere and collaborators [3, 4] both provide synthetic-logical accounts of differentiation. DiLL is motivated primarily by resource analysis in programming, and SDG grew out of the desire to give an "intrinsic" presentation of calculus and differential geometry in which concepts like smooth manifold and tangent vector are primitives instead of sophisticated derived concepts. While early accounts of SDG were formulated purely category theoretical in the language of toposes, the seminal work of Schreiber and Shulman in the last fifteen years [5, 6] – fueled by developments in **higher topos theory** [7] **and homotopy type theory** [8] **–** led to the emergence of **multi-modal type theories (MMTT)**, notably **cohesive homotopy type theory (CHoTT)** [10, 11] and the as of yet largely putative **differential-cohesive homotopy type theory (DCHoTT)** [12, 13] which promise to provide suitable frameworks for a proof theoretic account of SDG which is notably amenable to computer formalization in proof assistants such as Agda with its recently implemented *flat modality* [14].

Project: The general theme of the project is to pursue the study of DCHoTT and its models, and to explore analogies between DiLL and SDG/(D)CHoTT. These analogies start at the proof theoretic level, with Shulman's CHoTT incorporating split contexts and a promotion-like rule similar to – and inspired by – Benton's linear-non-linear logic [15], and continue on the semantic side where a central role is played by Taylor expansions both in DiLL [16] and SDG. A middle ground in comparing models of DiLL and models of DCHoTT is provided by **cartesian differential categories** [17, 18], but the relationship remains subtle. As a further step of clarification, we propose to investigate constructions of models of (D)CHoTT from models of DiLL, adapting ideas from **algebraic geometry**: extrapolating a point of view promoted by Melliès [19], we want to view the category of !-coalgebras of a model of linear logic as a category of **affine schemes**, which under suitable conditions (to be identified!) can be endowed with a notion of étale map and a Grothendieck topology by adapting techniques of Toën [20] and Lurie [21]. The linear structure is then reflected in the **induced sheaf topos** by a notion of **vector bundle**, and we expect there to be a well behaved 'tangent space' operation whenever the starting model of linear logic is differential. Besides a possible reconstruction of classical models of SDG [22], this approach provides a path to new models of SDG/DCHoTT based on recent work on 2-categorical models of DiLL [23].
As a possible parallel research avenue, the student could pursue a computer formalization project, which can serve as a means to familiarize them with the type-theoretic expression of fundamental concepts in differential geometry, and could also lead to a conference paper.

Relevance: The proposed research project lies at the intersection of cutting-edge areas in logic and theoretical computer science, which are also core research fields of the LoCal team at LIPN. Through the supervisors' international networks, the student will be integrated into a dynamic research community, which provides excellent opportunities for a future academic career.

References

[1] Ehrhard, Thomas, and Laurent Regnier. "The differential lambda-calculus." *Theoretical Computer Science* 309.1-3 (2003): 1-41.

[2] Ehrhard, Thomas, and Laurent Regnier. "Differential interaction nets." *Theoretical Computer Science* 364.2 (2006): 166-195.

[3] Lawvere, F. William. "Categorical dynamics." Topos theoretic methods in geometry 30 (1979): 1-28.

[4] Kock, Anders. *Synthetic differential geometry*. Vol. 333. Cambridge University Press, 2006.

[5] Schreiber, Urs. "Differential cohomology in a cohesive infinity-topos." *arXiv preprint arXiv:1310.7930* (2013).

[6] Schreiber, Urs, and Michael Shulman. "Quantum gauge field theory in cohesive homotopy type theory." *arXiv preprint arXiv:1408.0054* (2014).

[7] Lurie, Jacob. *Higher topos theory*. Princeton University Press, 2009.

[8] "Homotopy type theory: Univalent foundations of mathematics." *arXiv preprint arXiv:1308.0729* (2013).

[9] Gratzer, Daniel, et al. "Multimodal dependent type theory." *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*. 2020.

[10] Shulman, Michael. "Brouwer's fixed-point theorem in real-cohesive homotopy type theory." *Mathematical Structures in Computer Science* 28.6 (2018): 856-941.

[11] Frey, Jonas. "An introduction to Cohesive Homotopy Type Theory", Tutorial Talk at the 2022 NORTH AMERICAN ANNUAL MEETING OF THE ASSOCIATION FOR SYMBOLIC LOGIC, https://github.com/jonas-frey/pdfs/blob/master/asl-2022-slides.pdf

[12] Gross, Jacob A., et al. "Differential Cohesive Type Theory." *Extended abstracts for the Workshop" Homotopy Type Theory and Univalent Foundations*. 2017.

[13] Myers, David Jaz. "Orbifolds as microlinear types in synthetic differential cohesive homotopy type theory." *arXiv preprint arXiv:2205.15887* (2022).

[14] https://agda.readthedocs.io/en/latest/language/flat.html

[15] Benton, P. Nick. "A mixed linear and non-linear logic: Proofs, terms and models." International Workshop on Computer Science Logic. Berlin, Heidelberg: Springer Berlin Heidelberg, 1994.

[16] Mazza, Damiano. *Polyadic Approximations in Logic and Computation*. Diss. Université Paris 13, 2017.

[17] Blute, Richard F., J. Robin B. Cockett, and Robert AG Seely. "Cartesian differential categories." *Theory and Applications of Categories* 22.23 (2009): 622-672.

[18] Cruttwell, Geoff SH. "Cartesian differential categories revisited." *Mathematical structures in computer science* 27.1 (2017): 70-91.

[19] Melliès, Paul-André. "A Functorial Excursion Between Algebraic Geometry and Linear Logic." *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science*. 2022.

[20] Toën, Bertrand, and Michel Vaquié. "Au-dessous de Spec$\mathbb{Z}$." Journal of K-theory 3.3 (2009): 437-500.

[21] Lurie, Jacob. "Derived algebraic geometry V: Structured spaces." arXiv preprint arXiv:0905.0459 (2009)

[22] Moerdijk, Ieke, and Gonzalo E. Reyes. "Models for Smooth Infinitesimal Analysis (1991)."

[23] Marcelo Fiore, Nicola Gambino, and Martin Hyland. "Monoidal bicategories, differential linear logic, and analytic functors." *arXiv preprint arXiv:2405.05774* (2024).