Contrat Doctoral — ED Galilée

Titre du sujet : « Topology of triposes »

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- Domaine de recherche : Logique
- Mots-clés : Réalisabilité, Topos, Formalisation

Context

The notion of **tripos** was introduced in 1980 by Hyland, Johnstone, and Pitts [HJP80] as an auxiliary concept for the construction of *realizability toposes*, which are themselves categorical incarnations of Kleene's *number realizability* [Kle45] and are of interest as models of constructive logics and types theories (see e.g. [Reu99]). While Johnstone raised the question of an *axiomatization* of realizability in 2010 [Joh10] ('. . . what we lack is a general theory which would indicate just where the boundaries of the subject lie . . . '), more recently a consensus seems to have emerged that the notion of tripos itself does in fact provide such an axiomatization [Miq20; CMT21].

Since the class of triposes comprises in particular all *complete Heyting algebras* as 'forcing triposes', triposes can also be viewed as a generalization of the point-free spaces studied in *locale theory* [PP12], and the category of locales embeds contravariantly into the category of triposes and *regular tripos morphisms* considered in [Fre15b].

Project

The goal of the project is to study the category Trip of triposes and backwards regular tripos morphisms (or an 'enriched' version thereof in the style of [Fre14]) using methods from point-free topology [Joh83] and topos theory [Joh02]. Building on prior work [Fre13; Fre14; FS16; FS21], it is proposed to investigate the following:

(1) Examine closure properties of Trip under limits and colimits. In particular the existence of binary products would permit to consider the notion of 'triposic group', generalizing the notion of *localic group* [Isb+88; Joh88; Joh89] which is central to representation theorems in topos theory [BM87].
(2) Extend and systematize the study of factorization systems in Trip whose beginnings are sketched in [Fre14].
(3) Give a systematic account of *filter-quotients* of triposes. Filter-quotients are known from model theory [CK90] and in Trip admit a topological interpretation in terms of 'neighborhoods' – specifically the quotient of a localic tripos $\mathcal{P}$ at the neighborhood filter of a point $p \in \mathcal{P}$ can be viewed as the 'local neighborhood' of the point in the locale, a concept that resembles infinitesimal neighborhoods in algebraic geometry and cannot be captured with locales alone.
(4) Apply the above techniques to triposes constructed from *Krivine realizability interpretations* [Kri09], such as the triposes and constructions considered in [Fre15a; FS16].

Following the example of [CMT21; Coh+25], the student should formalize a part of their work in Rocq or another proof assistant. In particular the factorization construction (2) and accompanying definitions are sufficiently subtle to merit a mechanization.

Timeliness and impact

Realizability techniques and triposes have seen increased attention in recent years in set theory [Kar19; FM25] and computer science science [CMT21; Coh+25]. The proposed project further systematizes the study of these intriguing structures.

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