Contrat Doctoral — ED Galilée

Titre du sujet : « Algebraic K-Theory in simplicial Type Theory »

- Unité de recherche : LAGA et LIPN
- Discipline : Mathématiques, Informatique
- Direction de thèse : Christian AUSONI (co-directeur, LAGA), Jonas FREY (co-directeur, LIPN), Ulrik BUCHHOLTZ (co-encadrant externe, Université de Nottingham)
- Contact : ausoni@math.univ-paris13.fr, frey@lipn.fr, ulrik.buchholtz@nottingham.ac.uk
- Domaine de recherche : Théorie de l'homotopie stable, Théorie des Types
- Mots-clés : algebraic K-theory, simplicial Type Theory

Context

Simplicial Type Theory

Homotopy Type Theory (HoTT) [Uni13] emerged around twenty years ago, establishing a connection between the field of *abstract homotopy theory* in mathematics, and the field of *dependent type theory* which lies at the interface of computer science and formal logic. Dependent type theories are logical languages designed to formalize mathematical definitions and proofs, and HoTT is based on the striking observation that certain such theories are suitable to express homotopy theoretic concepts in a direct, 'synthetic' way, avoiding set theoretic 'boot-strapping'.

While HoTT has been successfully used to create libraries of computer-formalized developments of the foundations of abstract homotopy theory [Rij+; Agd], attempts to use 'pure' HoTT to give a synthetic account of the rapidly developing field of ∞-*category theory* [Lur09; Cis19] are confronted by fundamental obstructions. As a solution to this, Riehl and Shulman proposed **Simplicial Type Theory (STT)** [RS17], which augments HoTT by further primitives allowing to define ∞-categories in the style of Rezk's *complete Segal spaces* [Rez01]. This new field has seen substantial progress in recent years, with contributions notably by Buchholtz and collaborators [GWB24; GWB25; GWB26], and parallel formalization efforts [KRW24] in the newly developed proof assistant *Rzk* [Kud] going beyond prior work by Frey [Fre19] in *Lean*.

Algebraic K-theory

Algebraic K-theory is a fundamental invariant in different areas, including topology, number theory and (algebraic) geometry. The functor K_0 is known as Grothendieck's group, while higher algebraic K-theory is an extension of K_0 using tools from algebraic topology developed originally in the work of Quillen [Qui73] and Waldhausen [Wal85]. The rise of higher category theory revolutionized the subject, allowing for the characterization of algebraic K-theory and some of its variants by *universal properties* that were formerly mostly properties or slogans. What we have in mind here for example is the work of Blumberg Gepner and Tabuada [BGT13], characterizing the (connective) algebraic K-theory of small stable ∞-categories as the universal additive invariant. One application of this result sheds light on the nature of the Bökstedt trace map from algebraic K-theory to topological Hochschild homology, which has been a fundamental ingredient in the study of algebraic K-theory by Goodwillie's Calculus of Functors [Goo03].

Project

The principal goal of the project is to give synthetic presentations of fundamental concepts and results of stable homotopy and algebraic K-Theory in STT, showcasing the expressivity of this novel system and at the same time giving a more accessible and didactic account of the sophisticated mathematics. It is planned to pursue the investigation of some of following questions:

1. The construction and universal characterization of the ∞-category $\mathcal{S}p$ of *spectra* as the *stabilization* of the ∞-category $\mathcal{S}$ of *spaces*.
2. The definition of algebraic K-theory as a group completion of suitable monoidal categories (following the first steps taken by Buchholtz and collaborators [GWB26]), and its universal characterization in the style of Blumberg, Gepner, and Tabuada [BGT13].
3. The definition of topological Hochschild homology, for example in the commutative case, and the construction of the trace map.
4. The *Barratt–Priddy–Quillen theorem*, giving a universal characterization of the *sphere spectrum* as the algebraic K-theory of the groupoid of finite sets.

Depending on their inclination, the student may want to give computer formalizations of part of their arguments, or carry out the development purely on paper.

Impact

Type-theoretic computer formalization of mathematics has attracted growing interest from mainstream mathematics in recent years. Building on this momentum, the project combines state-of-the-art techniques from dependent type theory with advanced ideas from stable homotopy theory, developing new synthetic presentations of modern mathematical concepts that will shape how these subjects are taught in the future.

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